On the existence of a balanced vertex in geodesic nets with three boundary vertices

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Joint Mathematics Meetings

Geodesic Nets

A **geodesic** is a "straight" curve on a surface that can locally represent the shortest path between two points.

A **geodesic net** [3] is a network that connects multiple "unbalanced" points with the shortest curves while ensuring that each additional point is "balanced" and stretched equally by its neighbors through those curves.

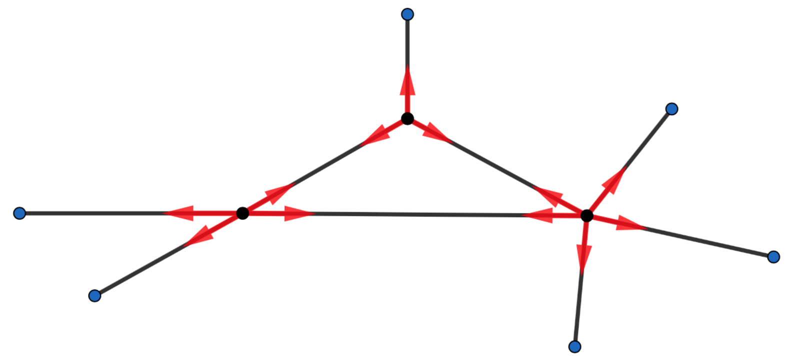


Figure 1. Geodesic net connecting 6 unbalanced points by 3 balanced points.

Question: How can we construct geodesic nets given any number of points?

Applications

Steiner Tree: a minimum-length network connecting points [2]

Telecommunication: design optimal network transporting data between centers

Molecular Biology: find the biological network to investigate the interaction among proteins and genes [1]

Urban Planning: determine the most efficient routes for public transportation systems

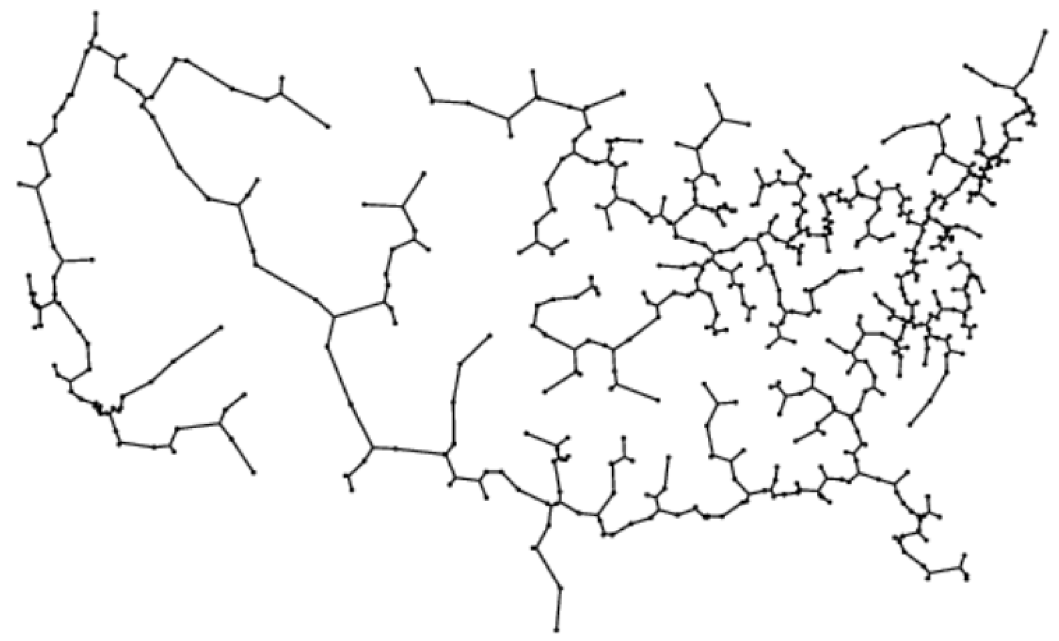
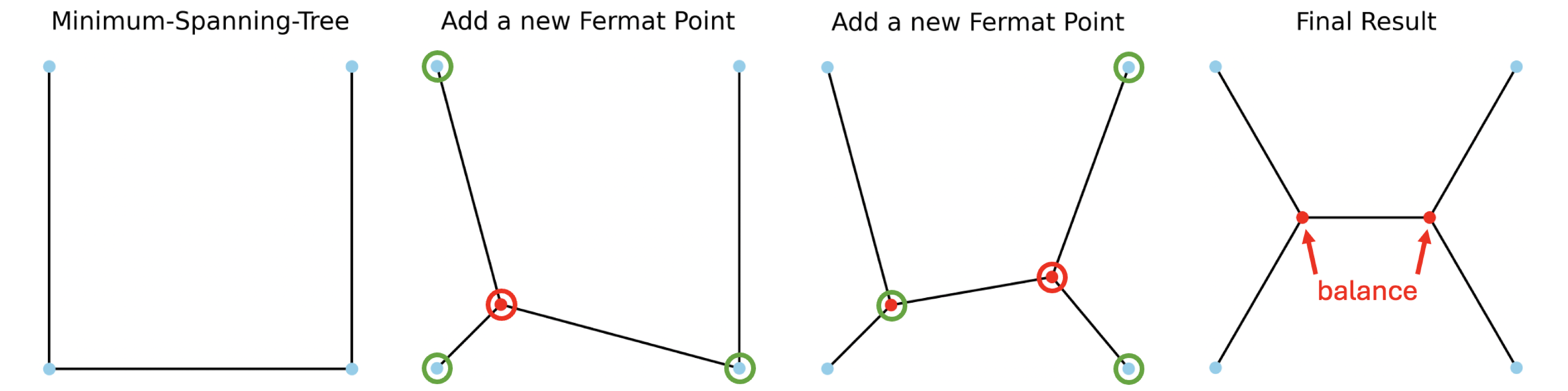
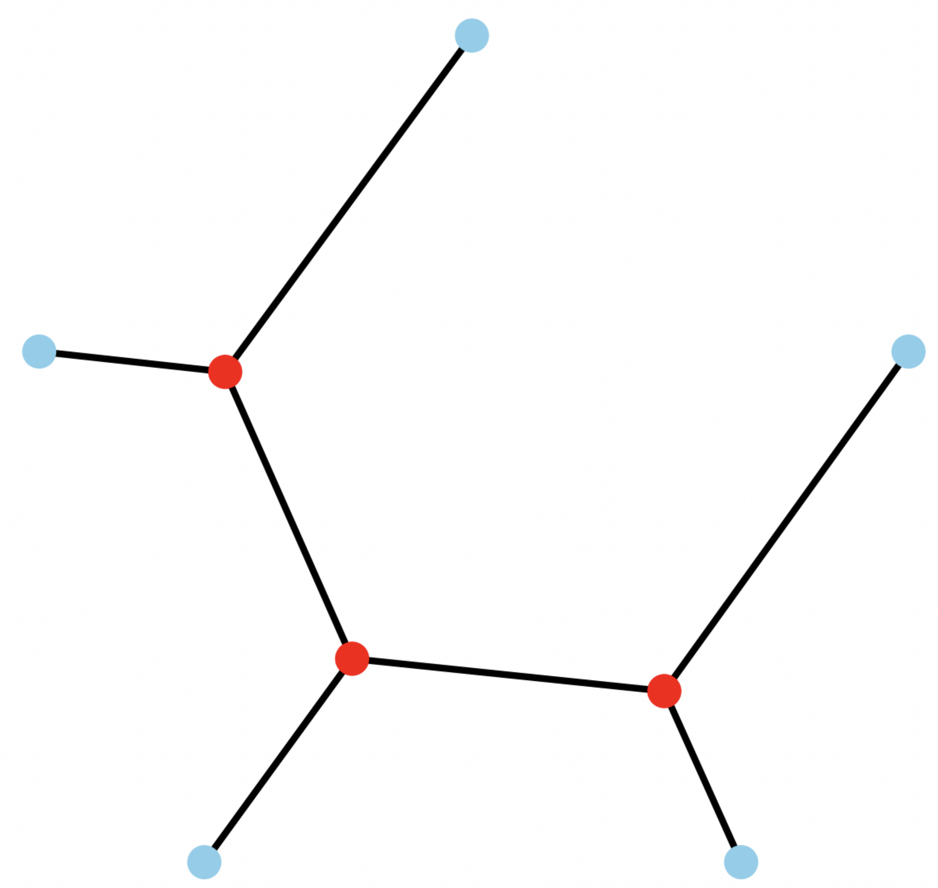


Figure 2. Steiner Tree of cities in the US [2].

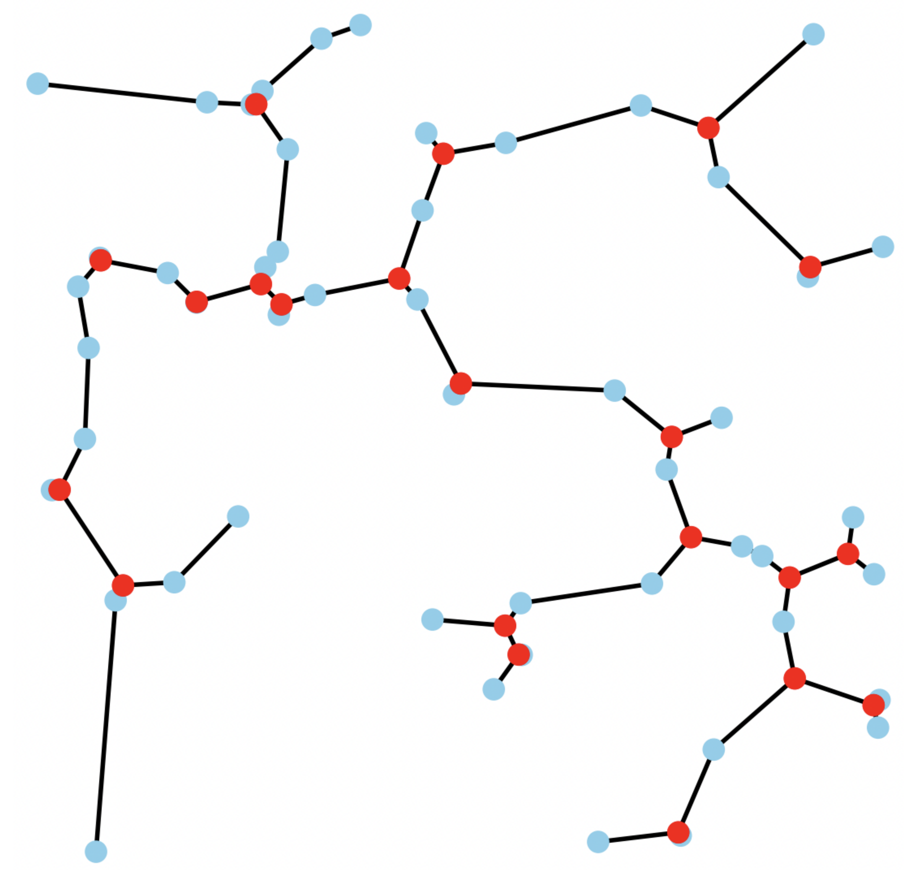
Construction Algorithm



Result with 5 points

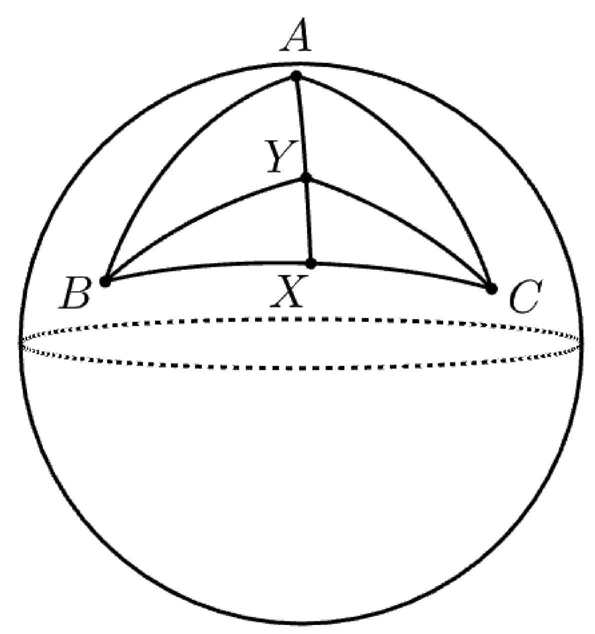


Result with 50 points

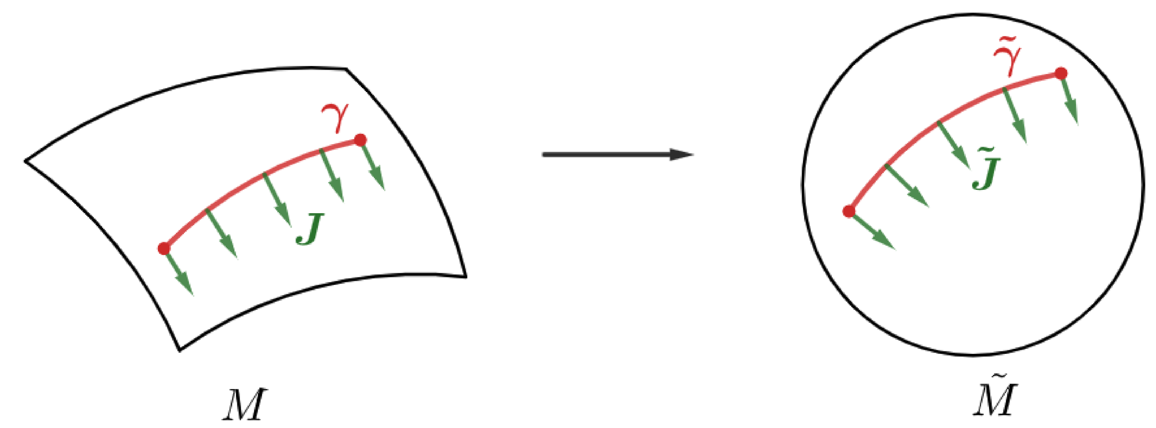


Theorems about the existence of a balanced vertex

Theorem 1: Given triangle ABC on a round sphere M with radius R such that its three angles measure less than $2\pi/3$. If the maximum geodesic distance of two points in the domain of the triangle ABC is less than $R\pi/2$, then there exists a balanced point.



Theorem 2: Let M be a Riemannian surface such that its Gaussian curvature K is bounded above by $1/R^2$, for $R > 0$. Let triangle ABC on M be given such that its three angles measure less than $2\pi/3$. If the maximum geodesic distance of two points in the domain of the triangle ABC is less than $R\pi/2$, then there exists a balanced point.



General Corollary

Corollary: On any surface with a Riemannian metric that is complete, there exists a neighborhood such that **Theorem 2** holds. More explicitly, the curvature on that neighborhood is bounded above by $1/R^2 > 0$, for $R > 0$. Thus, if a triangle ABC has three angles that measure less than $2\pi/3$ and the maximum geodesic distance of two points in the domain of the triangle is less than $R\pi/2$, then there exists a balanced point.

Future Works

- Implement the algorithm on different surfaces with different Riemannian metrics

- Investigate the complexity of the algorithm

References

[1] Nadja Betzler. "Steiner tree problems in the analysis of biological networks". In: *Masters thesis* (2006).

[2] Michael Herring. "The euclidean steiner tree problem". In: *Denison Univ., Granville, OH* (2004).

[3] Alexander Nabutovsky and Fabian Parsch. "Geodesic nets: Some examples and open problems". In: *Experimental Mathematics* 32.1 (2023), pp. 1–25.

Acknowledgement

This work is conducted under the supervision of Professor Ken Richardson and under Mathematics Departmental Honors program at Texas Christian University.

Website

Paper