Geodesic Nets
Construction and Existence

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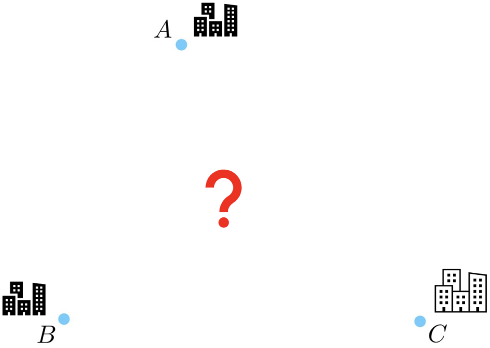
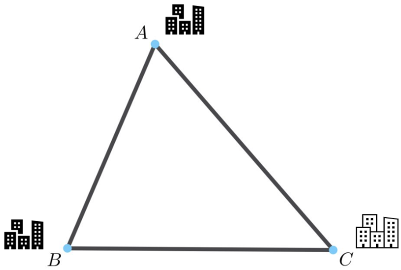
Motivation



Problem: Build a road system such that

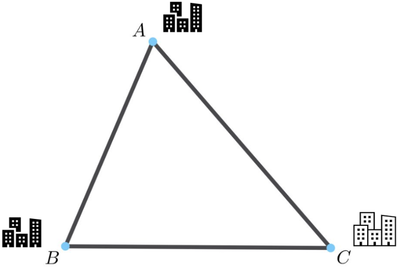
1. connecting all three cities
2. minimizing the total length of roads

Motivation



(a) Connect all cities

Motivation



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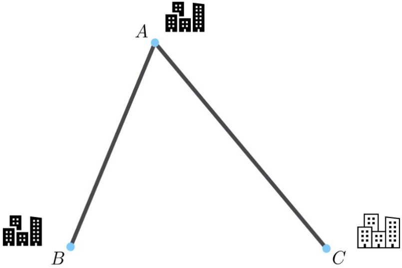
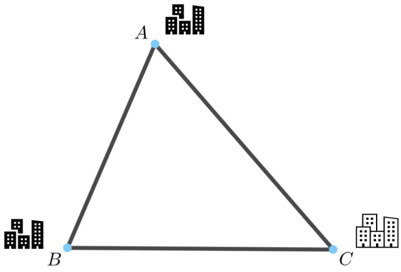


(b) Connect all cities + Smaller total length

Motivation



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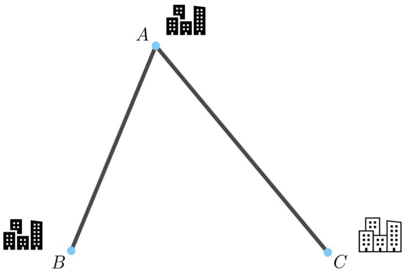

(b) Connect all cities + Smaller total length

But does this network minimize the total length?

Fermat Point

The **Fermat point** F of triangle ABC is the point that **minimizes** the **sum of three distances** from three vertices to the point.

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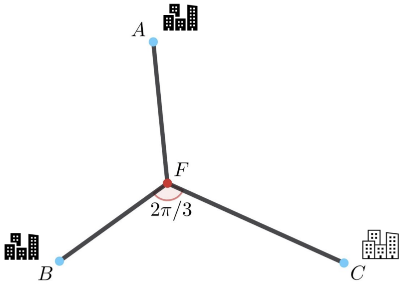

(a) Case 1: all angles $< 2\pi/3$

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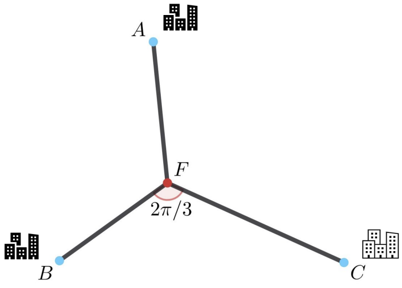

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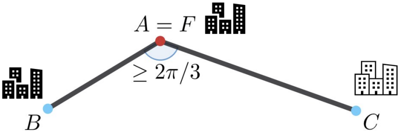

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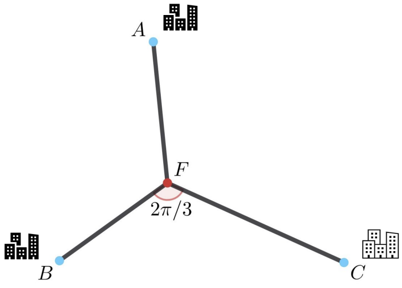

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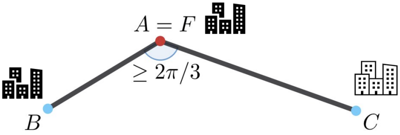

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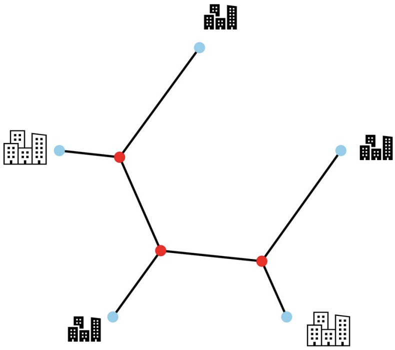
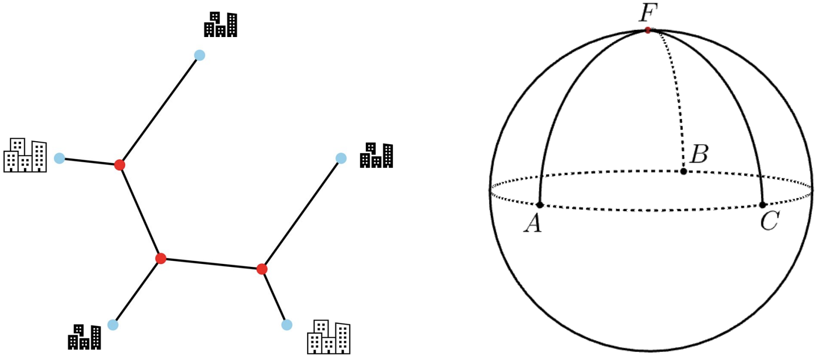
Important property: in Case 1, $\angle AFB = \angle BFC = \angle CFA = 2\pi/3$.

General problems



(a) **Steiner Tree** [2] of *n points*

General problems



(a) **Steiner Tree** [2] of *n* points (b) Networks on different surfaces

Applications

- **Telecommunication:** design optimal network transporting data between centers
- **Molecular Biology:** find the biological network to investigate the interaction among proteins and genes [1]
- **Urban Planning:** determine the most efficient routes for public transportation systems

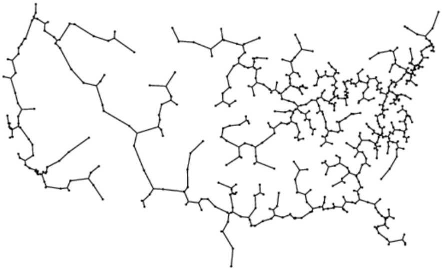
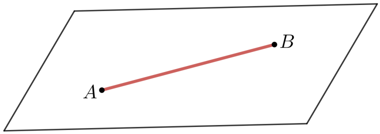


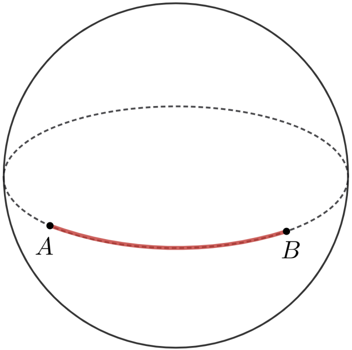
Figure: Steiner Tree of cities in the US [2].

Geodesics

A **geodesic** is a "straight" curve on a surface.



(a) Straight line on a plane



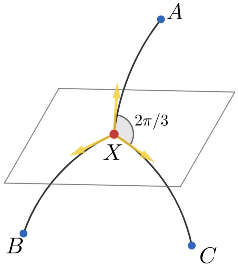
(b) Geodesic on a sphere

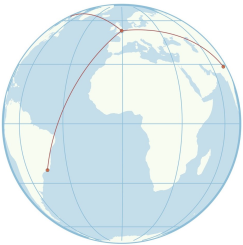
Important property: Geodesic locally minimizes length.

Geodesic Nets

A **geodesic net** [3] is a **network** connecting points with **geodesics** on surfaces that includes:

- "unbalanced" points.
- "balanced" points - stretched equally.





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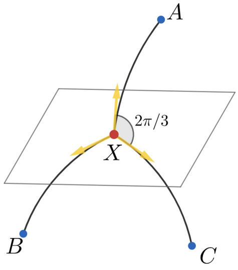

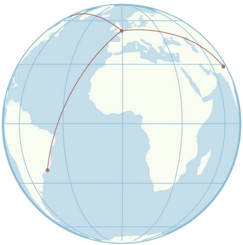

Question: How to **construct** geodesic nets given **any number of points**?

Big picture

Main idea: Given n unbalanced points, add new balanced points heuristically.

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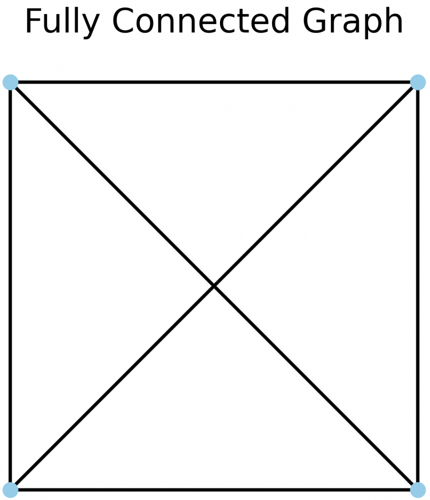
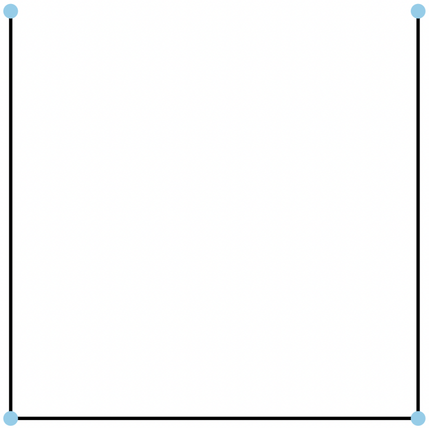
If each unbalanced point v_u has degree 1 or connects to two other points with an angle greater than or equal to $2\pi/3$, the algorithm terminates. Otherwise, the process continues by adding a balanced point.

Algorithm Visualization



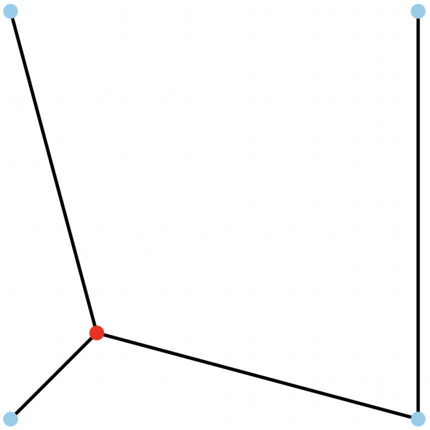
Algorithm Visualization

Minimum-Spanning-Tree

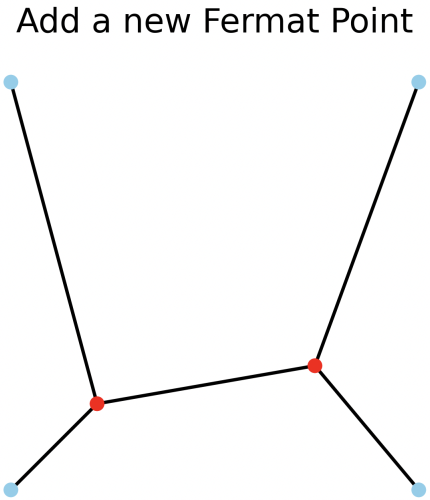


Algorithm Visualization

Add a new Fermat Point

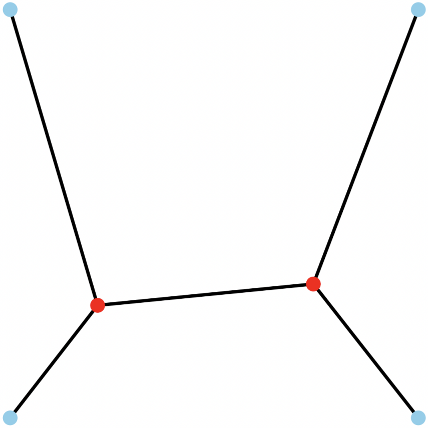


Algorithm Visualization

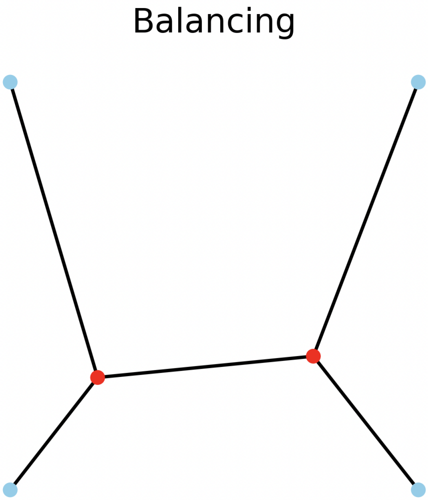


Algorithm Visualization

Balancing



Algorithm Visualization



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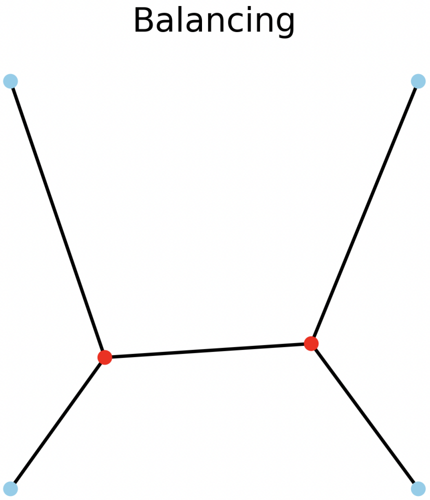

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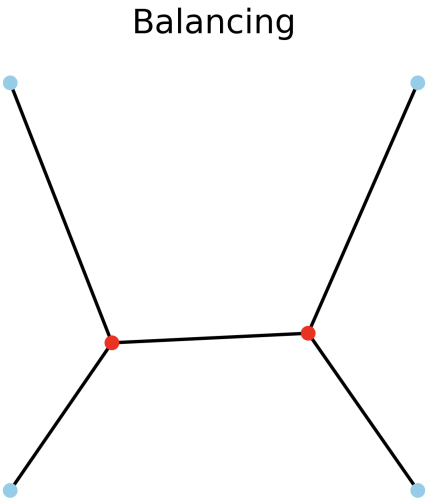

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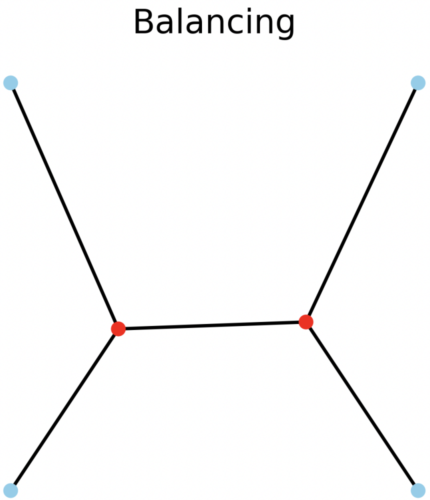

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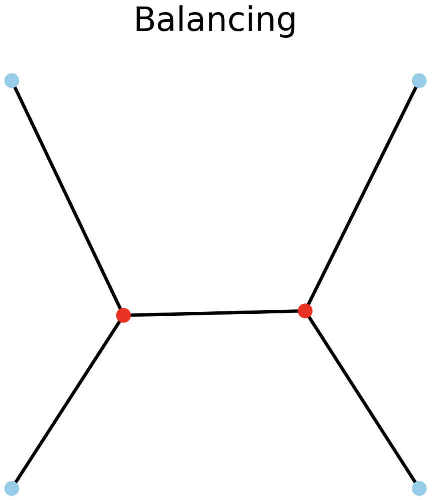

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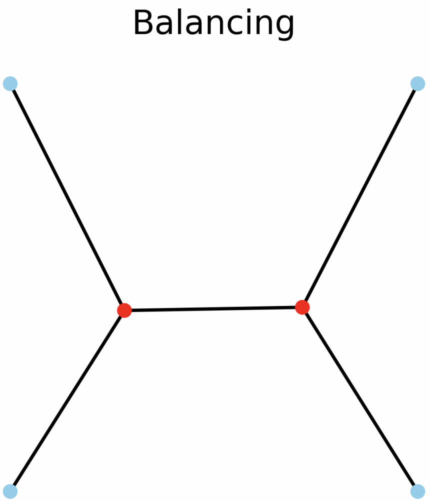

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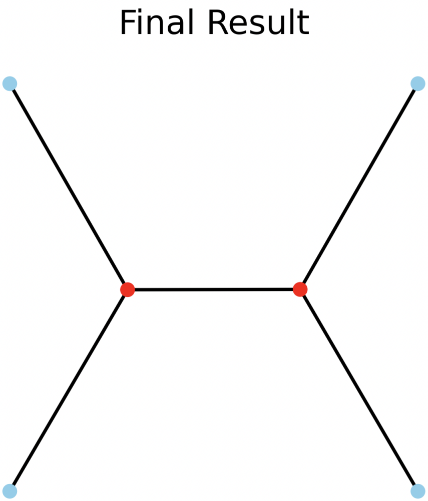
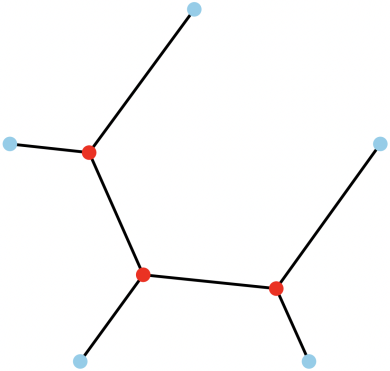
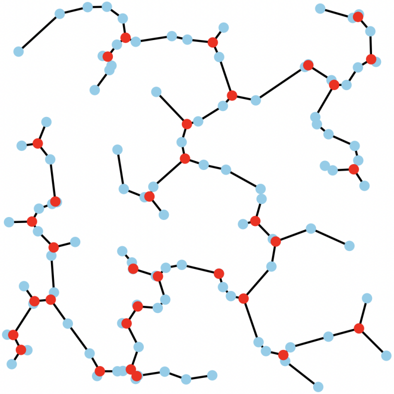


Results



(a) 5 unbalanced points

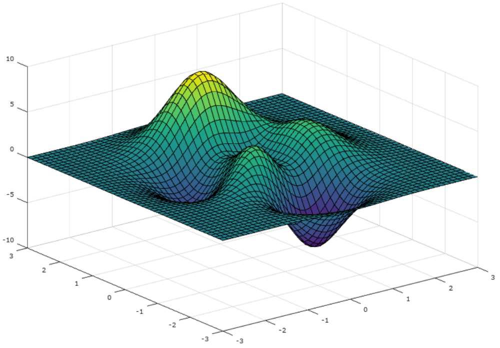


(b) 100 unbalanced points

Motivation

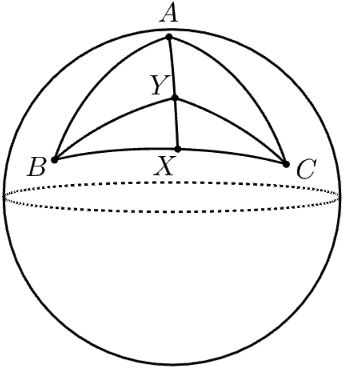
Recall that the main idea of the algorithm is adding new balanced points.

Question: Do balanced points exist on a general surface?



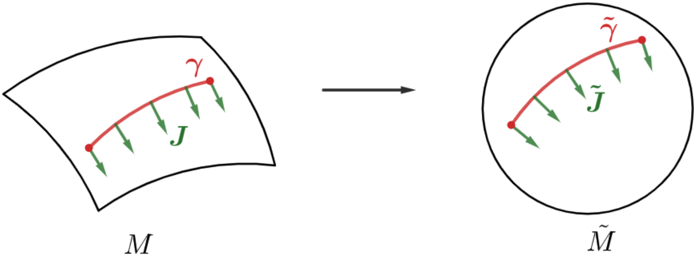
Existence of a balanced vertex - Sphere

Theorem 1 (*Sphere*). Given triangle ABC on a round sphere M with radius R such that its three angles measure less than $2\pi/3$. If the maximum geodesic distance of two points in the domain of the triangle ABC is less than $R\pi/2$, then **there exists a balanced point**.



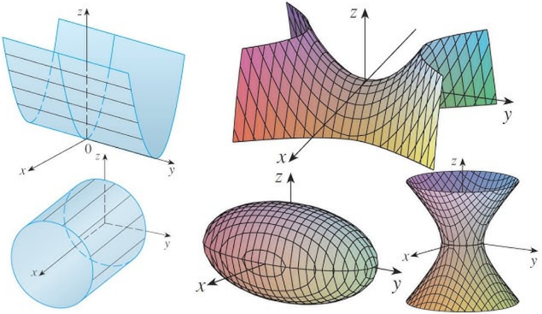
Existence of a balanced vertex - General surface

Theorem 2 (*General surface*). Let M be a Riemannian surface such that its Gaussian curvature K is **bounded above** by $1/R^2$, for $R > 0$. Let triangle ABC on M be given such that its three angles measure less than $2\pi/3$. If the maximum geodesic distance of two points in the domain of the triangle ABC is less than $R\pi/2$, then **there exists a balanced point**.



Future work

- Implement the algorithm on **different surfaces** with **different Riemannian metrics**
- Investigate the **complexity** of the algorithm



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The euclidean steiner tree problem.
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Geodesic nets: Some examples and open problems.
Experimental Mathematics, 32(1):1–25, 2023.

Ending

(a) Thesis

(b) Paper

Nguyen, Duc Toan. "On the existence of a balanced vertex in geodesic nets with three boundary vertices." *Journal of Geometry* 116.3 (2025): 36.